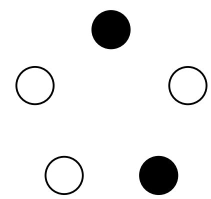
CEMC Mathematics Teachers' Conference
Dinner Time Math

Welcome to Dinner Time Math! We have selected a number of challenging problems for your mathematical enjoyment. Work with the other guests sitting at your table to find as many correct answers as possible. The table answering the greatest number of questions correctly before time runs out (we'll announce when that is) will receive a prize! Good luck!

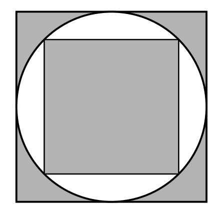
1. Three white candles and two black candles can be arranged in a number of ways in a pentagon shaped candelabra.

   If the candles are placed at random, find the probability that the three white candles will be adjacent.

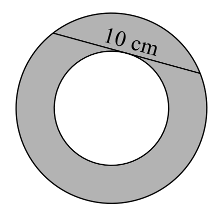


2. A white circle is inscribed in the larger grey square and the smaller grey square is inscribed in the circle.

   What fraction of the diagram is white?



3. The circles in the diagram are concentric (they have the same centre). Find the area of the annulus (shaded area).



4. In a particular class, each student has blonde hair or brown eyes; 1/4 of students with blonde hair have brown eyes and 1/3 of the students with brown eyes have blonde hair. What fraction of the total class have brown eyes?

5. The Fibonacci sequence is defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, where $F_1 = 1$ and $F_2 = 1$.

   $$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

   Consider the ratio of adjacent terms, $\dfrac{F_{n+1}}{F_n}$:

$1/1 = 1$	$2/1 = 2$	$3/2 = 1.5$	$5/3 = 1.666\ldots$	$8/5 = 1.6$
$13/8 = 1.625$	$21/13 = 1.615\ldots$	$34/21 = 1.619\ldots$	$55/34 = 1.617\ldots$	$89/55 = 1.618\ldots$

   As n increases it is well known that the ratio of adjacent terms $\dfrac{F_{n+1}}{F_n}$ tends towards

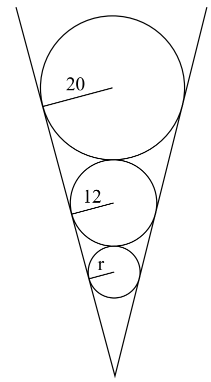
   $$\phi = \frac{\sqrt{5}+1}{2}$$

   What does $\dfrac{F_{n+2}}{F_n}$ tend towards as n increases?

6. For all real numbers, $|x|$ is defined as the absolute value of x. For example, $|4.2| = 4.2$ and $|-7| = 7$. Given that x and y are integers, how many different solutions does the equation $|x| + 2|y| = 100$ have?

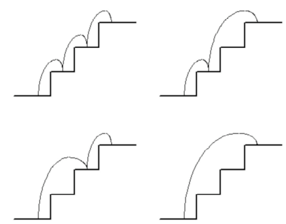
7. Three balls are placed inside a cone such that each ball is in contact with the edge of the cone and the next ball.

   If the radii of the balls are 20, 12, and r respectively, what is the value of r?

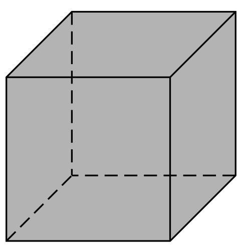


8. It is possible to climb three steps in exactly four different ways.

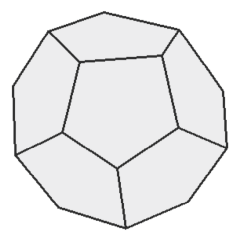
   How many ways can you climb ten steps?



9. A cube (regular hexahedron) consists of 6 faces and 12 edges.



A dodecahedron has 12 faces. How many edges does it have?



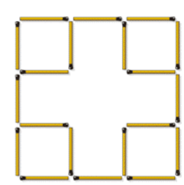
10. What is the smallest positive 4 digit number that is a multiple of each of the numbers 1 through 10?

11. Consider the infinite sequence:

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$$

What is the 1000$^{\text{th}}$ term?

12. The matchsticks currently form 5 squares. Move two matchsticks so that there are 7 squares. Every matchstick must be a part of at least one square, and you may not break matchsticks. Draw the resulting figure.



13. A confused bank teller transposed the dollars and cents when he cashed a cheque for Ms. Smith, giving her dollars instead of cents and cents instead of dollars. After buying a newspaper for 50 cents, Ms Smith noticed that she had left exactly three times as much as the original cheque. What was the amount of the cheque? (Note: 1 dollar = 100 cents.)

14. A rectangular sheet of paper is folded so that two diagonally opposite corners come together. If the crease formed is the same length as the longer side of the sheet, what is the ratio of the longer side of the sheet to the shorter side?

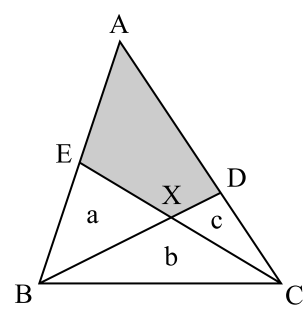
15. A 12 cm by 25 cm by 36 cm cereal box is lying on the floor on one of its 25 cm by 36 cm faces. An ant, located at one of the bottom corners of the box, must crawl along the outside of the box to reach the opposite bottom corner. What is the length of the shortest such path?

    Note: The ant can walk on any of the five faces of the box, except for the bottom face, which is flush in contact with the floor. It can crawl along any of the edges. It cannot crawl under the box.

16. The sum of the reciprocals of two real numbers is -1, and the sum of the cubes of the two numbers is 4. What are the numbers?

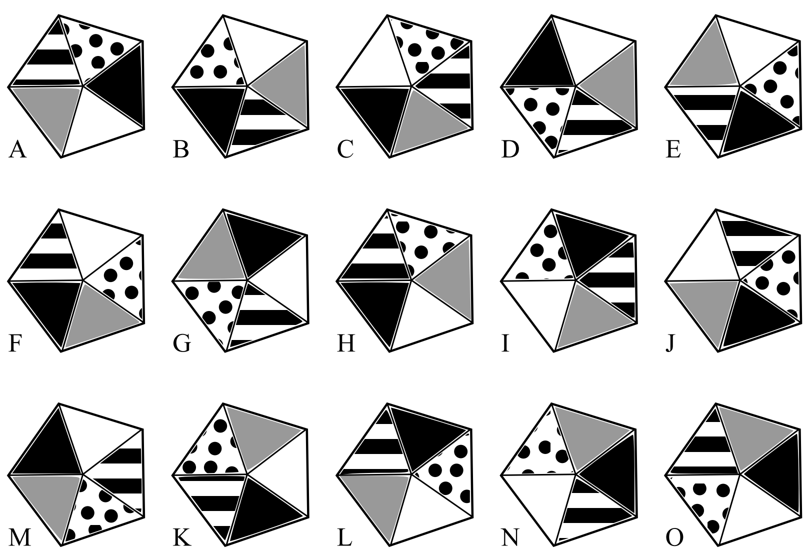
17. In $\triangle ABC$:

    - Produce a line from B to AC, meeting at D.
    - Produce a line from C to AB, meeting at E.
    - Let BD and CE meet at X.
    - Let $\triangle BXE$ have area a.
    - Let $\triangle BXC$ have area b.
    - Let $\triangle CXD$ have area c.

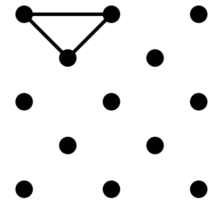


    Find the area of quadrilateral $AEXD$ in terms of a, b, and c.

18. Find the pentagon without an identical partner.

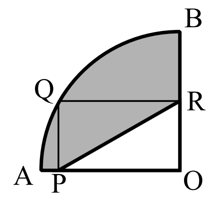


19. How many right isosceles triangles of any size and orientation can you find in the shape using the given points as vertices? One triangle is already shown.

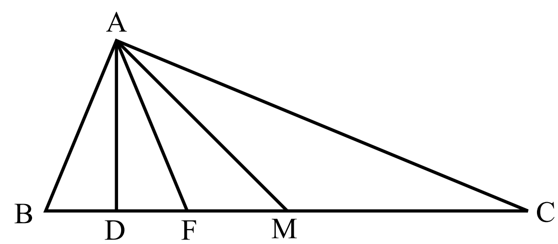


20. In your high school, there are 73 lockers along a wall. One student goes along and opens every locker. A second student goes along and closes every second locker beginning with locker number 2. A third student changes the state of every third locker beginning with locker number 3 (If it's open, he closes it. If it's closed, he opens it.) A fourth student goes along and performs the same action with every fourth locker, and so on, until 73 students have gone along. How many lockers are open after all 73 students have passed by?

21. In the diagram, AOB is a quarter circle of radius 10 and $PQRO$ is a rectangle of perimeter 26. What is the perimeter of the shaded region?



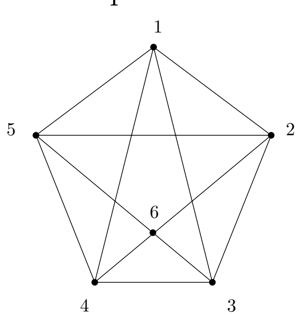
22. In $\triangle ABC$ shown, the altitude AD, the angle bisector AF and the median AM divide $\angle BAC$ into four equal parts. Calculate the angles of the triangle.



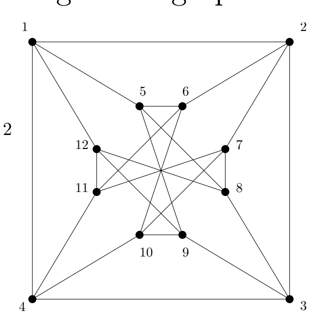
23. Positive integers a, b and c satisfy $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c^2}$.

What is the sum of all possible values of $a \leq 100$?

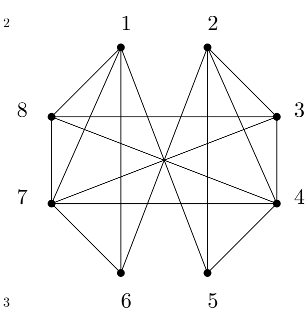
24. A graph is a collection of vertices with pairs of vertices connected by edges. A planar graph is a graph that CAN be drawn without edges crossing. Such a drawing is called a planar embedding. In other words, if we are given a graph with edges crossing, it may still be planar if there is a way to rearrange the points so that no edges cross. State whether each of the graphs below is PLANAR or NOT PLANAR. If it is planar, draw a planar embedding of the graph.



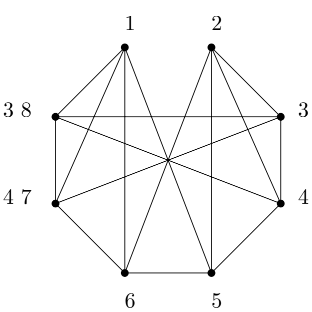
a



b



c



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